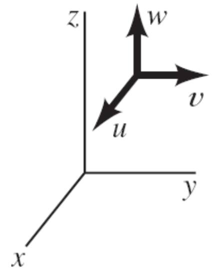
Viscous flow



Equations of motion in cartesian coordinates:

x-direction
$$\underbrace{\rho(\frac{\partial u}{\partial t}}_{\text{Local Accel.}} + \underbrace{\text{u}\frac{\partial u}{\partial x} + \text{v}\frac{\partial u}{\partial y} + \text{w}\frac{\partial u}{\partial z}}_{\text{Convective Acceleration}}) = \underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z})}_{\text{Surface forces}}$$

y-direction
$$\rho(\frac{\partial v}{\partial t} + \text{u}\frac{\partial v}{\partial x} + \text{v}\frac{\partial v}{\partial y} + \text{w}\frac{\partial v}{\partial z}) = \rho g_y + (\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z})$$

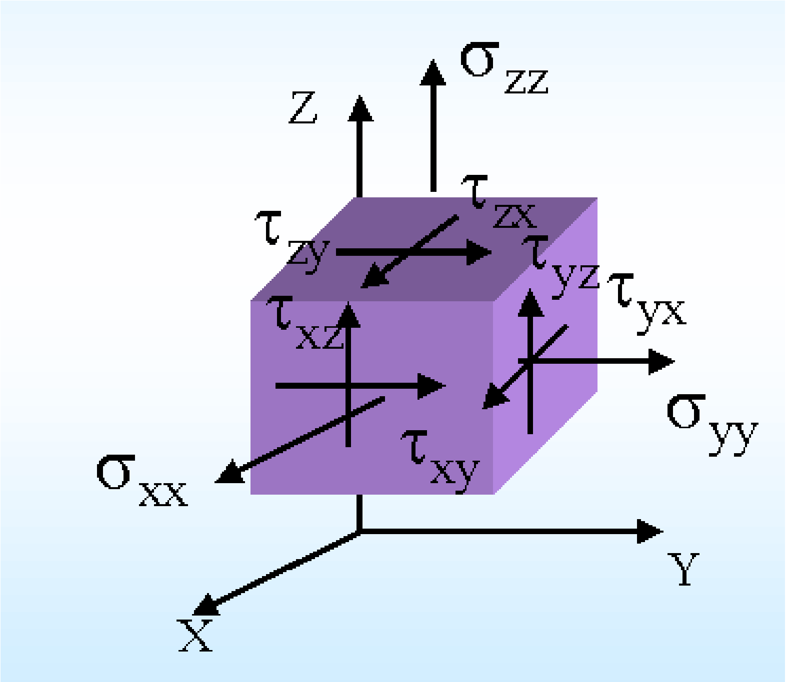
z-direction
$$\rho(\frac{\partial w}{\partial t} + \text{u}\frac{\partial w}{\partial x} + \text{v}\frac{\partial w}{\partial y} + \text{w}\frac{\partial w}{\partial z}) = \rho g_z + (\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})$$

The equations of motion include stresses (σ_{ij}) and velocities (*u*, *v* and *w*)

We need a relationship between stresses (σ_{ij}) and velocities (u, v and w)

Stress-deformation relationships

For Newtonian, incompressible fluids, stresses are linearly related to deformations



Normal stresses:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

Shearing stresses:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Eliminate stresses

In x-direction: $\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \rho g_x + (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z})$

$$\sigma_{xx} = -p+2\mu\frac{\partial u}{\partial x} \longrightarrow \frac{\partial \sigma_{xx}}{\partial x} = -\frac{\partial p}{\partial x} +2\mu\frac{\partial^2 u}{\partial x^2}$$

$$\tau_{xy} = \tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \longrightarrow \frac{\partial \tau_{yx}}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x \partial y}$$

$$\tau_{zx} = \tau_{xz} = \mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) \longrightarrow \frac{\partial \tau_{zx}}{\partial z} = \mu\frac{\partial^2 w}{\partial x \partial z} + \mu\frac{\partial^2 u}{\partial z^2}$$

The right hand term of the equation becomes:

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} +2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial v}{\partial x \partial y} + \mu\frac{\partial w}{\partial x \partial z} + \mu\frac{\partial^2 u}{\partial z^2}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x \partial y} + \frac{\partial w}{\partial x \partial z})$$

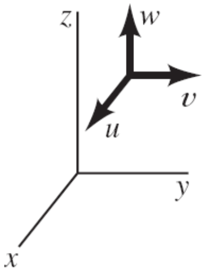
$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\underbrace{(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})}_{=0\text{ : continuity equation}}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Navier-Stokes equations (cartesian)

$$\left.\begin{aligned}
&\text{x-direction:} \quad \underbrace{\rho(\frac{\partial u}{\partial t}}_{\substack{\text{Local}\\\text{Accel.}}} + \underbrace{\mathrm{u}\frac{\partial u}{\partial x} + \mathrm{v}\frac{\partial u}{\partial y} + \mathrm{w}\frac{\partial u}{\partial z})}_{\substack{\text{Convective}\\\text{Acceleration}}} = \underbrace{-\frac{\partial p}{\partial x}}_{\text{Pressure}} + \underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\substack{\text{Viscous}\\\text{forces}}} \\
&\text{y-direction:} \quad \rho(\frac{\partial v}{\partial t} + \mathrm{u}\frac{\partial v}{\partial x} + \mathrm{v}\frac{\partial v}{\partial y} + \mathrm{w}\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
&\text{z-direction:} \quad \rho(\frac{\partial w}{\partial t} + \mathrm{u}\frac{\partial w}{\partial x} + \mathrm{v}\frac{\partial w}{\partial y} + \mathrm{w}\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}\right\}\ \text{Momentum equations}$$

$$+$$



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Continuity equation

Stress-deformation relationships for an incompressible fluid (cylindrical)

Normal stresses:

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\mu (\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r})$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

Shearing stresses:

$$\tau_{r\theta} = \tau_{\theta r} = \mu (r \frac{\partial}{\partial r} (\frac{v_\theta}{r}) + \frac{1}{r} \frac{\partial v_r}{\partial \theta})$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu (\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta})$$

$$\tau_{zr} = \tau_{rz} = \mu (\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z})$$

Navier-Stokes equations (cylindrical)

r, θ and z momentum equations:

r:

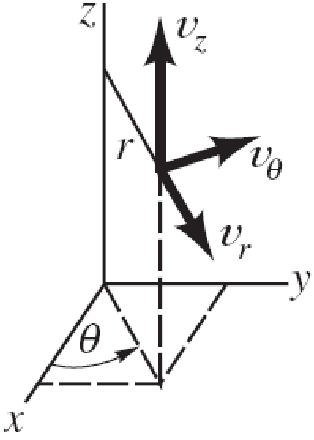
$$\rho(\underbrace{\frac{\partial v_r}{\partial t}}_{\text{Local Accel.}}+\underbrace{v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}}_{\text{Convective Acceleration}})=\underbrace{-\frac{\partial p}{\partial r}}_{\text{Pressure}}+\underbrace{\rho g_r}_{\text{Gravity}}+\mu(\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_r}{\partial r})-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}}_{\text{Viscous forces}})$$

θ:

$$\rho(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z})=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_\theta}{\partial r})-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2})$$

z:

$$\rho(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z})=-\frac{\partial p}{\partial z}+\rho g_z+\mu(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r})+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2})$$



+

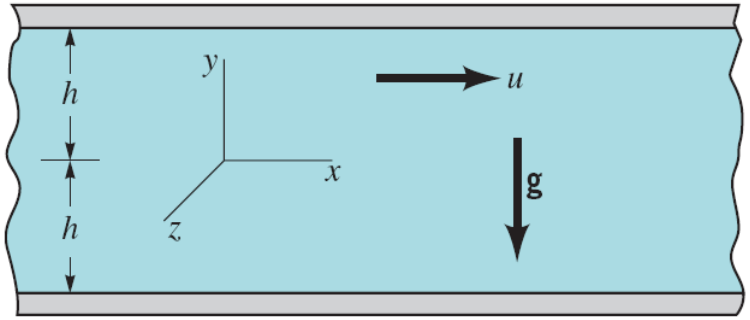
Continuity equation:

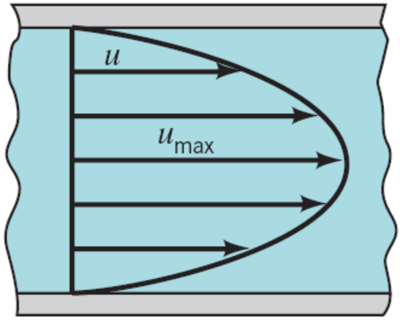
$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Simple solutions for viscous, incompressible fluids

- Principal difficulty: nonlinearities from the convective acceleration terms.
- Exact solution exist only for few cases

1) Steady, laminar flow between infinite parallel plates



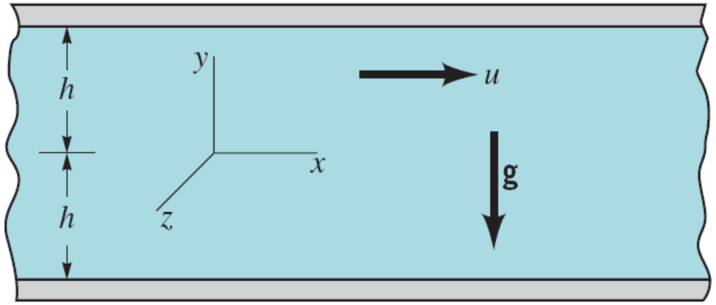


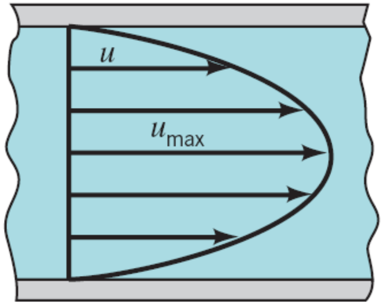
$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+u\cancel{\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}+\cancel{w\frac{\partial u}{\partial z}}\right)=-\frac{\partial p}{\partial x}+\cancel{\rho g_x}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\rho\left(\cancel{\frac{\partial v}{\partial t}}+u\cancel{\frac{\partial v}{\partial x}}+v\cancel{\frac{\partial v}{\partial y}}+w\cancel{\frac{\partial v}{\partial z}}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}+\cancel{\frac{\partial^2 v}{\partial y^2}}+\cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

$$\rho\left(\cancel{\frac{\partial w}{\partial t}}+u\cancel{\frac{\partial w}{\partial x}}+v\cancel{\frac{\partial w}{\partial y}}+w\cancel{\frac{\partial w}{\partial z}}\right)=-\frac{\partial p}{\partial z}+\cancel{\rho g_z}+\mu\left(\cancel{\frac{\partial^2 w}{\partial x^2}}+\cancel{\frac{\partial^2 w}{\partial y^2}}+\cancel{\frac{\partial^2 w}{\partial z^2}}\right)$$

Steady, laminar flow between infinite parallel plates (cont'd)





$$x:\quad 0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}\qquad (1)$$

$$y:\quad 0=-\frac{\partial p}{\partial y}-\rho g\qquad (2)$$

$$z:\quad 0=-\frac{\partial p}{\partial z}\qquad (3)$$

Boundary conditions:

u(h)=u(-h)=0

$(2) \Rightarrow p = -\rho g y + f_1(x)$ (hydrostatic variation in y)

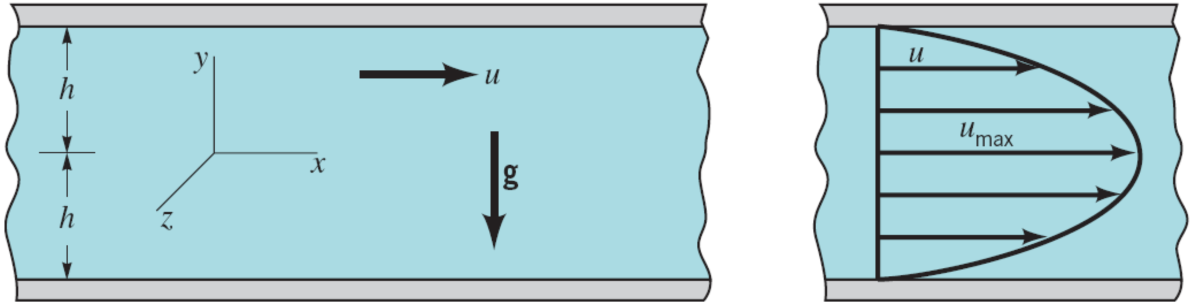
not a function of y

$$(1) \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \Rightarrow \frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + c_1$$

$$\Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + c_1 y + c_2$$

from B.C.

Steady, laminar flow between infinite parallel plates (cont'd)



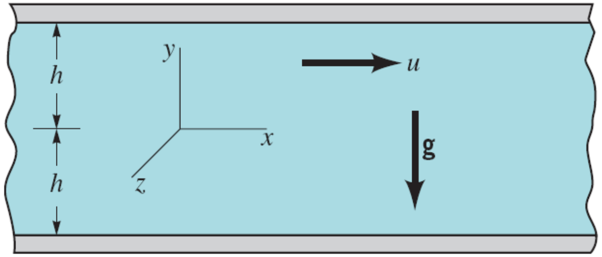
$u = 0$ for $y = h \Rightarrow 0 = \frac{1}{2\mu}\frac{\partial P}{\partial x} h^2 + C_1 h + C_2$

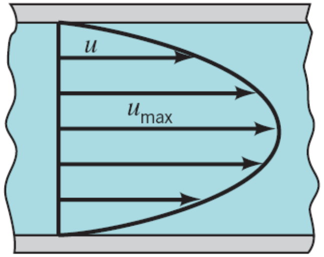
$u = 0$ for $y = -h \Rightarrow 0 = \frac{1}{2\mu}\frac{\partial P}{\partial x} h^2 - C_1 h + C_2$

$\Rightarrow \quad C_1 = 0$ and $C_2 = -\frac{1}{2\mu}\frac{\partial P}{\partial x} h^2$

$$\therefore \quad \underline{\underline{u = -\frac{1}{2\mu}\frac{\partial P}{\partial x}\left(h^2 - y^2\right)}}$$

Steady, laminar flow between infinite parallel plates (cont'd)





Volume rate of flow, q (per unit width in z):

$$q=\int_{-h}^{h} u\cdot dy = \int_{-h}^{h} -\frac{1}{2\mu}\frac{\partial p}{\partial x}(h^2-y^2)\,dy = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[h^2y-\frac{y^3}{3}\right]_{-h}^{h}$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[h^3-\frac{h^3}{3}-\left(-h^3+\frac{h^3}{3}\right)\right] = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[2h^3-\frac{2h^3}{3}\right] = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x}$$

$$\therefore \quad \underline{\underline{q = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x}}}$$

The pressure gradient is negative: $\underline{-\frac{\partial p}{\partial x} = \frac{\Delta p}{\ell}}$

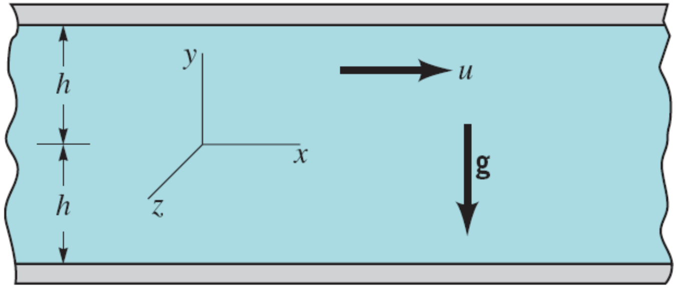
where $\frac{\Delta p}{\ell}$ is the <u>pressure drop per unit length</u>

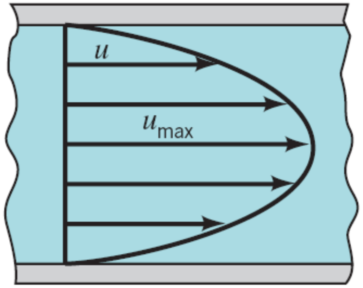
$$\therefore \quad \underline{\underline{q = \frac{2h^3}{3\mu}\frac{\Delta p}{\ell}}}$$

Average velocity: $\underline{\underline{\overline{V} = \frac{q}{2h} = \frac{h^2}{3\mu}\frac{\Delta p}{\ell}}}$

Maximum velocity: $\underline{\underline{V_{max} = \frac{1}{2\mu}\frac{\Delta p}{\ell}h^2 = \frac{3}{2}\overline{V}}}$

Steady, laminar flow between infinite parallel plates (cont'd)





For pressures:

$$(2) \Rightarrow P = -\rho g y + f_1(x)$$

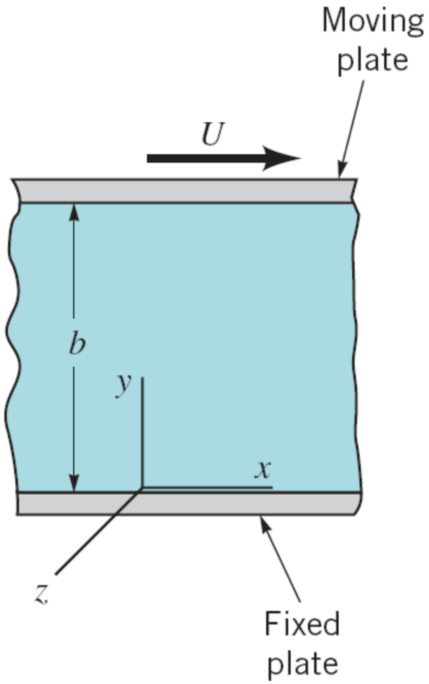
$$\Rightarrow \frac{\partial P}{\partial x} = \frac{df_1}{dx} = ct$$

$$\Rightarrow f(x) = \frac{\partial P}{\partial x} x + C$$

$$\left.\begin{aligned} &\therefore \quad P(x,y) = -\rho g y + \frac{\partial P}{\partial x} x + C \\ &\text{Let} \quad P(x=0, y=0) = P_0 \quad \Rightarrow \quad C = P_0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \underline{\underline{P(x,y) = P_0 - \rho g y + \frac{\partial P}{\partial x} x}}$$

Couette flow



The integration of x-momentum yields:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2 \qquad (1)$$

B.C.: $y = 0 \rightsquigarrow u = 0 \Rightarrow C_2 = 0$

$y = b \rightsquigarrow u = U$

$$\Rightarrow U = \frac{1}{2\mu}\frac{\partial p}{\partial x} b^2 + C_1 b$$

$$\Rightarrow C_1 = \frac{U}{b} - \frac{1}{2\mu}\frac{\partial p}{\partial x} b \qquad (2)$$

$$(1) \Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + \frac{U}{b} y - \frac{1}{2\mu}\frac{\partial p}{\partial x} b y$$

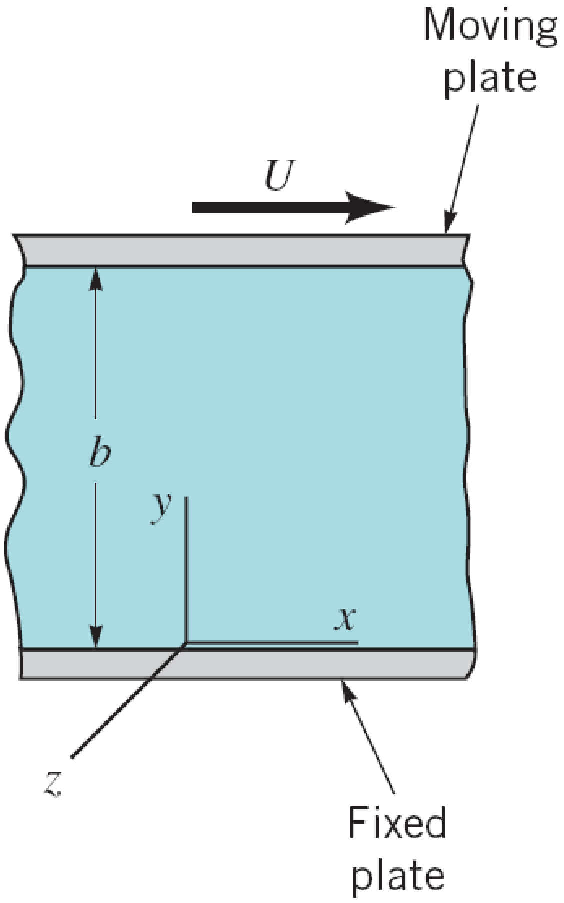
$$\Rightarrow u = \frac{U}{b} y + \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - b y\right)$$

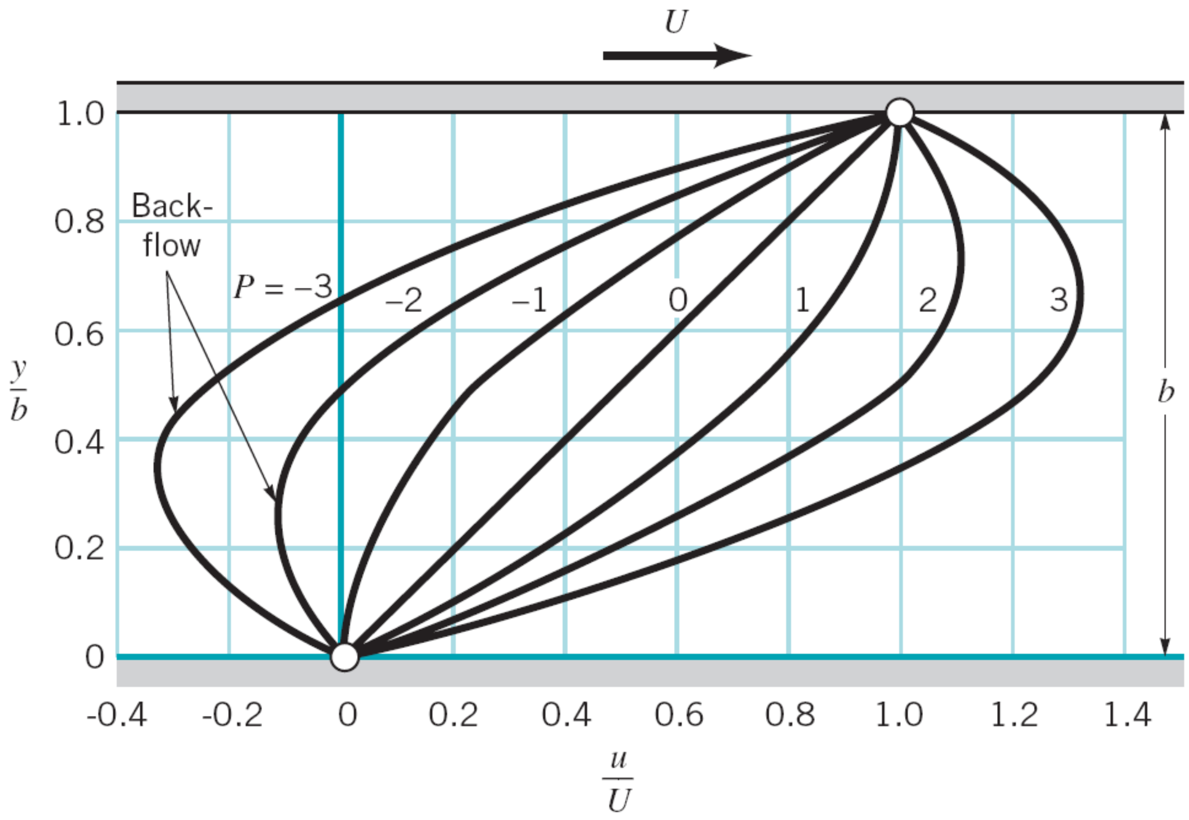
or, in non-dimensional form:

$$\underline{\underline{\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\frac{\partial p}{\partial x}\frac{y}{b}\left(1 - \frac{y}{b}\right)}}$$

Let $P = -\frac{b^2}{2\mu U}\frac{\partial p}{\partial x}$

Couette flow

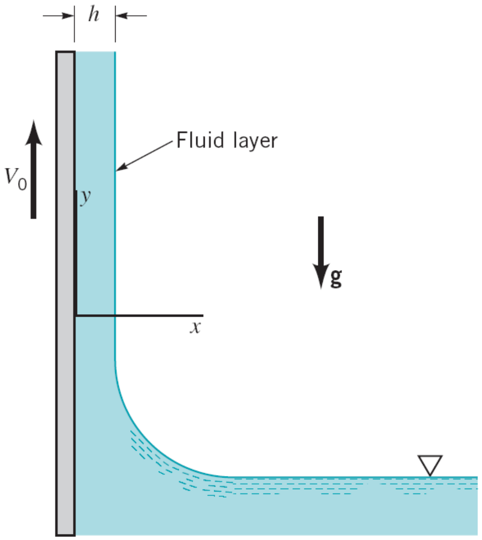




$$\underline{\underline{\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\frac{\partial p}{\partial x}\frac{y}{b}\left(1 - \frac{y}{b}\right)}}$$

$$P = -\frac{b^2}{2\mu U}\frac{\partial p}{\partial x}$$

Example on plane Couette flow



Belt moving vertically upward with constant velocity V_0, is dragging along a viscous fluid from a reservoir.

Find an expression for the average velocity of the fluid layer.

The only velocity component is v $(u = w = 0)$

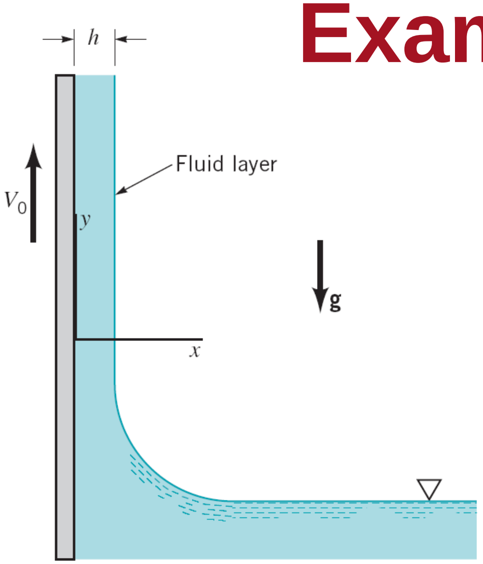
From continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow \underline{\underline{v = v(x)}}$

$$\left.\begin{aligned} &x\text{-momentum}: \frac{\partial P}{\partial x} = 0 \\ &z\text{-momentum}: \frac{\partial P}{\partial z} = 0 \end{aligned}\right\} \Rightarrow$$

Pressure does not vary in the horizontal plane

Since $P = 0$ at the film interface with air (atmospheric) $\Rightarrow$ $\underline{\underline{P=0}}$ $\underline{\text{everywhere in the film}}$

Example on plane Couette flow



$$y\text{-momentum: } 0 = -\rho g + \mu \frac{d^2 v}{dx^2} \Rightarrow \frac{d^2 v}{dx^2} = \frac{\gamma}{\mu} \quad (1)$$

$$\text{Integrate (1)} \Rightarrow \frac{dv}{dx} = \frac{\gamma}{\mu} x + c_1 \quad (2)$$

At the interface with air ($x=h$), the shear stress is zero (air drag is negligible)

$$\tau_{x=h} = \mu \frac{dv}{dx}\Big|_{x=h} = 0 \overset{(2)}{\Longrightarrow} \frac{\gamma}{\mu} h + c_1 = 0$$

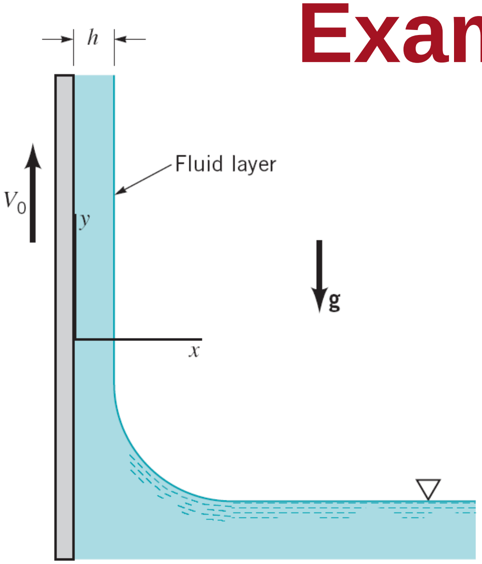
$$\Rightarrow C_1 = -\frac{\gamma h}{\mu} \quad (3)$$

$$\therefore \frac{dv}{dx} = \frac{\gamma}{\mu} x - \frac{\gamma h}{\mu} \Rightarrow v(x) = \frac{1}{2}\frac{\gamma}{\mu} x^2 - \frac{\gamma h}{\mu} x + c_2$$

$$\text{B.C. } v = V_0 \text{ @ } x = 0 \Rightarrow C_2 = V_0$$

$$\text{Hence, } \underline{\underline{v(x) = \frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + V_0}} \quad (4)$$

Example on plane Couette flow



The flowrate per unit width, q is:

$$q = \int_0^h v(x)\,dx = \int_0^h \left(\frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0\right)dx$$

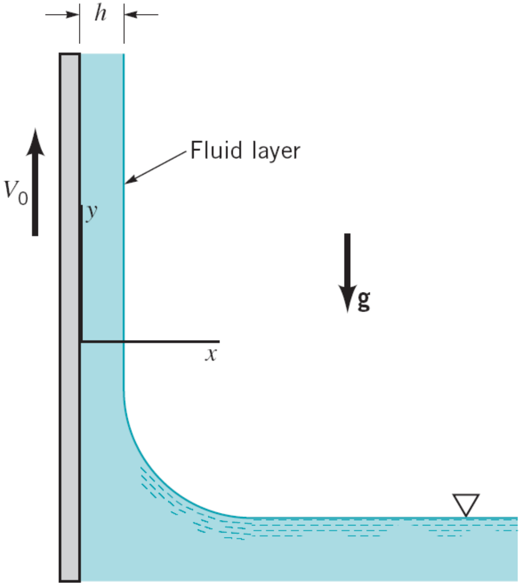
$$= \left[\frac{\gamma x^3}{6\mu} - \frac{\gamma h x^2}{2\mu} + V_0 x\right]_0^h = \frac{\gamma h^3}{6\mu} - \frac{\gamma h^3}{2\mu} + V_0 h$$

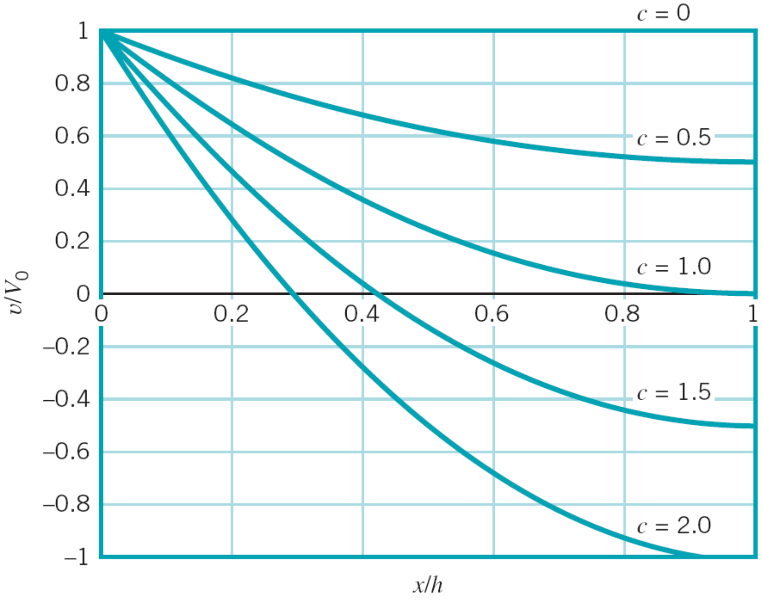
$$\Rightarrow \quad \underline{\underline{q = V_0 h - \frac{\gamma h^3}{3\mu}}}$$

The average velocity is:

$$\underline{\underline{\bar{V} = \frac{q}{h} = V_0 - \frac{\gamma h^2}{3\mu}}}$$

Example on plane Couette flow



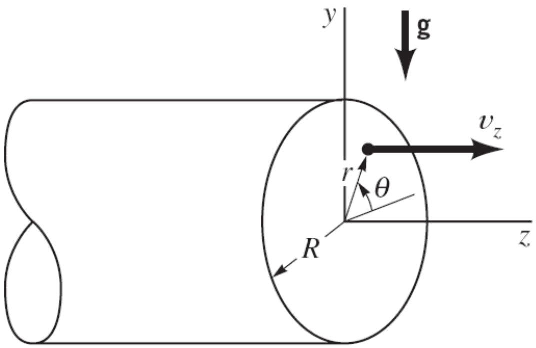


In non-dimensional form:

$$\frac{v}{V_0} = \frac{\gamma h^2}{2\mu V_0}\left(\frac{x}{h}\right)^2 - \frac{\gamma h^2}{\mu V_0}\left(\frac{x}{h}\right) + 1 \qquad (5)$$

$$\text{Let} \quad c = \frac{\gamma h^2}{2\mu V_0} \Rightarrow \underline{\underline{\frac{v}{V_0} = c\left(\frac{x}{h}\right)^2 - 2c\left(\frac{x}{h}\right) + 1 \qquad (6)}}$$

Steady, laminar flow in a horizontal tube: (Hagen-Poiseuille)



Momentum in z: $0 = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial P}{\partial z} \qquad \left(\frac{\partial P}{\partial z} = ct\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial P}{\partial z}r$$

$$\Rightarrow r\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\frac{\partial P}{\partial z}r^2 + C_1 \Rightarrow \frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\frac{\partial P}{\partial z}r + \frac{C_1}{r}$$

$$\Rightarrow v_z = \frac{1}{4\mu}\frac{\partial P}{\partial z}r^2 + C_1 \ln r + C_2 \qquad (1)$$

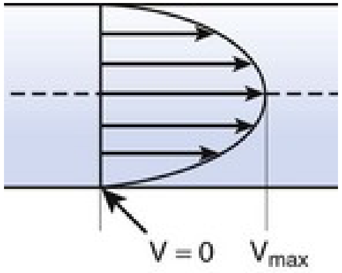
Boundary conditions:

v_z is finite @ $r=0$ $\Rightarrow C_1 = 0$ (2)

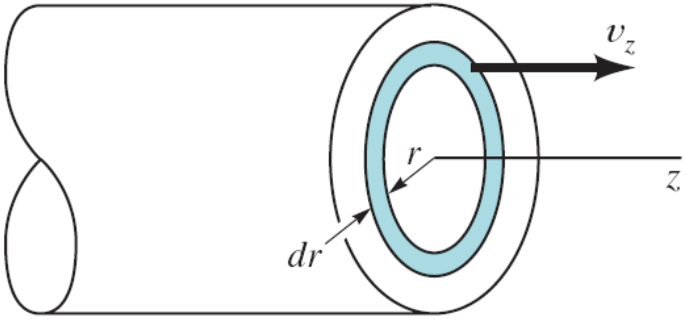
$v_z = 0$ @ $r=R$ $\Rightarrow C_2 = -\frac{1}{4\mu}\frac{\partial P}{\partial z}R^2$ (3)

$$\therefore v_z(r) = -\frac{1}{4\mu}\frac{\partial P}{\partial z}\left(R^2 - r^2\right)$$

parabolic profile



Poiseuille's law



$$dQ = V_z\, dA = V_z\,(2\pi r \cdot dr)$$

$$Q = \int_0^R V_z\, 2\pi r\, dr = -2\pi \frac{1}{4\mu}\frac{\partial P}{\partial z}\int_0^R (R^2 - r^2)\, r\, dr$$

$$= -\frac{\pi}{2\mu}\frac{\partial P}{\partial z}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R$$

$$= -\frac{\pi}{2\mu}\frac{\partial P}{\partial z}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = -\frac{\pi R^4}{8\mu}\frac{\partial P}{\partial z}$$

Pressure drop per unit length $\frac{\Delta p}{\ell} = -\frac{\partial P}{\partial z}$

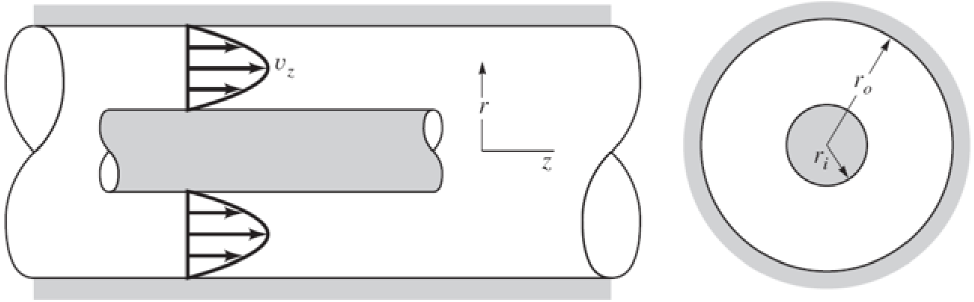
$$\therefore \quad Q = \frac{\pi R^4}{8\mu}\frac{\Delta p}{\ell}$$ Poiseuille's law

Average velocity: $\overline{V} = \frac{Q}{\pi R^2} = \frac{R^2}{8\mu}\frac{\Delta p}{\ell}$

Flow is laminar for $Re < 2100$
where $Re = \frac{\rho V D}{\mu}$

$$V_{max} = -\frac{1}{4\mu}\frac{\partial P}{\partial z} R^2 = 2\overline{V}$$

Steady, axial, laminar flow in an annulus



Derive expressions for:
a) the axial velocity profile, *u(r)*
b) the flow rate, *Q*

Assuming developed flow, the x-momentum equation of the Navier-Stokes equations is given by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

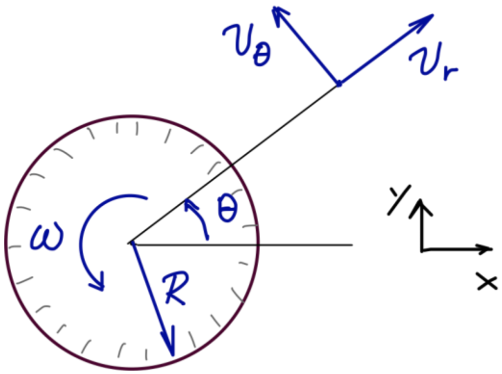
Integrating twice we obtain $v_z = \frac{1}{4\mu}\frac{\partial p}{\partial z}r^2 + C_1 \ln r + C_2$

Applying the boundary conditions u(r=r$_i$) = u(r=r$_o$) = 0, we come up with the final expression for the velocity distribution:

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o/r_i)}\ln\frac{r}{r_o}\right]$$

The flow Q can then be derived by simple integration: $Q = \int_{r_i}^{r_o} v_z(2\pi r)\,dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$

Example: flow around a rotating cylinder



Infinitely long, vertical cylinder turning with rotational speed ω.

Derive an expression for the velocity distribution.

For the particular flow field $v_r = 0$; $v_z = 0$

Continuity in cylindical coordinates:

$$\frac{1}{r}\frac{\partial(r\cancelto{0}{v_r})}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial \cancelto{0}{v_z}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial v_\theta}{\partial \theta} = 0$$

The Navier-Stokes equation in the θ-direction:

$$\rho\left(\cancelto{0}{\frac{\partial v_\theta}{\partial t}} + \cancelto{0}{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}} + \frac{\cancelto{0}{v_r} v_\theta}{r} + \cancelto{0}{v_z}\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho\cancelto{0}{g_\theta} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\cancelto{0}{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{2}{r^2}\cancelto{0}{\frac{\partial v_r}{\partial \theta}} + \cancelto{0}{\frac{\partial^2 v_\theta}{\partial z^2}}\right)$$

$$\Rightarrow \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}\right]$$

Due to axisymmetry: $\frac{\partial p}{\partial \theta} = 0$

so that $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0$$

$$\Rightarrow \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} = 0 \qquad (1)$$

Since v_θ is a function of r only:

$$\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0 \qquad (2)$$

Integrating once, we obtain:

$$\frac{dv_\theta}{dr} + \frac{v_\theta}{r} = c_1 \quad \Rightarrow \quad r\frac{dv_\theta}{dr} + v_\theta = c_1 \cdot r$$

$$\Rightarrow \quad \frac{d(r v_\theta)}{dr} = c_1 \cdot r \quad \Rightarrow \quad r v_\theta = \frac{1}{2} c_1 r^2 + c_2$$

$$\Rightarrow \qquad v_\theta = \frac{1}{2} c_1 r + \frac{c_2}{r}$$

B.C. : @ $r = \infty \quad v_\theta = 0 \Rightarrow c_1 = 0$

@ $r = R \quad v_\theta = \omega R \Rightarrow c_2 = \omega R^2$

Finally: $$\underline{\underline{v_\theta = \frac{\omega R^2}{r}}}$$